SMALL PERTURBATIONS SPECTRUM IN BOUNDARY-LAYER FLOW ON A FLAT PLATE

O. A. Likhachev

UDC 532.516

The small perturbation spectrum of a number of flows has recently been analyzed carefully [1-3]. At the same time, investigations for the boundary layer have been limited within the framework of linear perturbation theory to the neighborhood of the neutral curve although a spectrum analysis is of indubitable interest not only to find the stability criterion of a laminar stream, but also to solve a problem with initial data about the time development of an arbitrary small perturbation. In particular, the possibility of representing an arbitrary perturbation in terms of a system of basis functions is related to the question of the completeness of the system. The finiteness was proved [4] and an estimate was obtained of the domain of eigenvalue existence in an investigation of the boundary-layer stability and a deduction has been made about the finiteness of the small perturbations spectrum for boundary-layer flow on this basis. A sufficiently complete survey of the investigation of the neutral stability of a laminar boundary layer can be found in the monograph [5]. The small perturbations spectrum in a boundary layer flow is obtained in this paper by methods of the linear theory of hydrodynamic stability by using the complete boundary conditions on the outer boundary. It is shown that the small perturbations spectrum is finite for each fixed value of the wave number α. Singularities in the spectrum behavior are investigated for sufficiently small α.

It is usually assumed in investigations of boundary-layer stability that the perturbation on the outer stream boundary damps out as a solution of the inviscid problem

$$\varphi \sim e^{-\alpha y}.$$

The complete conditions for damping of the perturbations at infinity which were first formulated in [6] are considered in this paper:

$$\begin{gathered}(\varphi''-\alpha^2\varphi)'+\gamma(\varphi''-\alpha^2\varphi)=0;\\ (\varphi''-\gamma^2\varphi)'+\alpha(\varphi''-\gamma^2\varphi)=0 \text{ for } y=\delta.\end{gathered} \tag{1}$$

Here $\gamma=\gamma_r+i\gamma_e$,

$$\gamma_r=\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}},\quad \gamma_e=\frac{b}{2\gamma_r},\quad a=\alpha^2+\alpha RY,$$
$$b=\alpha\,\mathrm{Re}\,(1-X),$$

where C = X + iY is the desired eigenvalue ($Y<0$ corresponds to exponential damping of the perturbations), and δ is the boundary-layer thickness. The prime denotes the derivative with respect to y.

The boundary-layer equations admit of self-similar solutions for flows around a flat plate, and affine profiles are obtained by the introduction of similarity transformations for the independent variable and the stream function ψ in the form [7]

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 112-115, July-August, 1975. Original article submitted November 25, 1974.

$$\eta = y\sqrt{\frac{U}{\nu x}};\quad \psi = \sqrt{\nu x U}\cdot f(\eta).$$

This similarity transformation reduces the system of boundary-layer equations to an ordinary differential equation,

$$2f''' + ff'' = 0,$$

$f = f' = 0$ for $\eta = 0$; $f' = 1$ for $\eta = \delta$, where the prime here denotes the derivative with respect to η. The longitudinal velocity component hence equals

$$u = \frac{1}{U}\frac{\partial\psi}{\partial y} = f',\quad 0 \leqslant \eta \leqslant \delta. \tag{2}$$

The assumption that the flow in the boundary layer is plane-parallel with the profile (2) is henceforth used.

The hydrodynamic stability problem of plane-parallel viscous incompressible fluid flow reduces to an analysis of the eigenvalue spectrum of the Orr—Sommerfeld equation.

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi - i\alpha\mathrm{Re}\,[(u - C)(\varphi'' - \alpha^2\varphi) - u''\varphi] = 0$$

with adhesion conditions on the wall $\varphi(0) = \varphi'(0) = 0$ and conditions (1) on the outer boundary. It is hence considered that the velocity is constant and equal to the potential flow velocity U outside the boundary layer for $\eta > \delta$. The perturbations damp out at infinity if $\gamma_r > 0$. Here two-dimensional perturbations are considered the most dangerous, since the Squire theorem is valid in this case.

Let us examine the case when the phase velocity X is close to one so that $|b| \ll |a|$, in greater detail. Then if only positive γ_r is admitted, the continuous continuation of the boundary conditions as X passes through one is possible only for $a > 0$. If $a < 0$, as is actually realized for sufficiently small values of the wave number α, then the boundary conditions will vary by a jump for the passage through X = 1 since γ_e becomes discontinuous; in the neighborhood of X = 1

$$\gamma_r = \frac{|b|}{2\sqrt{|a|}};\quad \gamma_e = \mathrm{sign}\, b\sqrt{|a|}.$$

Therefore, the spectrum of the damped perturbations cannot be continued continuously in terms of the wave number at which X = 1. Let us call such a wave number α_* limiting. For $\alpha < \alpha_*$ the spectrum for perturbations damping out at infinity apparently simply does not exist.

If the constraint $\gamma_r > 0$ is removed, then the continuous passage through α_* can be accomplished upon selecting the branches

$$\gamma_r = \frac{b}{2\sqrt{|a|}};\quad \gamma_e = \sqrt{|a|}.$$

This corresponds to the fact that the perturbations damp out at infinity for $X < 1$ and grow for $X > 1$. Hence, it was assumed that

$$\gamma_r = \frac{b}{2\gamma_e};\quad \gamma_e = \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}},$$

but it should be recalled that perturbations with $X > 1$ have no physical meaning. The problem was solved numerically by using the method proposed in [8].

The search for the spectrum was carried out by the method of making a transition in a parameter from the known spectrum for channel flow. On the outer limit of the boundary-layer the conditions for the perturbations were posed in the form

$$\begin{pmatrix}\varphi\\ \varphi'\end{pmatrix} = \begin{pmatrix}A^2 + B & AB\\ A & B\end{pmatrix}\begin{pmatrix}\varphi''\\ \varphi'''\end{pmatrix}\ (1 - \varepsilon),$$

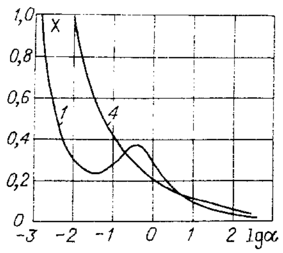

Fig. 1

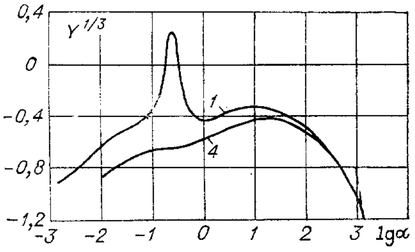

Fig. 2

where $A=-(\gamma+\alpha)/\alpha\gamma$; $B=-1/\alpha\gamma$; ε is the parameter to which conditions (1) correspond for $\varepsilon=0$ and the adhesion conditions correspond for $\varepsilon=1$. An asymptotic expression is known for the spectrum [2] in a channel for $\alpha\ll1$, which can be used to construct the whole small-perturbations spectrum by a continuous transition in α. Then performing a continuous transition in the parameter ε for a fixed value of the wave number α, we realize the passage from the adhesion conditions at the outer boundary to the perturbation damping conditions at infinity. The perturbations in a channel are divided into two classes for $\alpha\gg1$: near the wall for which $X\to0$, and near the axis with the phase velocity $X\to1$. The perturbations have the phase velocity for all values of the wave number α, after the passage in the parameter ε, for the near-axis modes, i.e., the modes obtained do not satisfy the physical requirement of perturbation damping at infinity and should be excluded from consideration. Therefore, there are no shortwave perturbations localized near the outer flow boundary in the boundary layer. The numbering of the spectrum modes for boundary-layer flow corresponds to conservation of the number of the spectrum mode from which the transition is accomplished.

Represented in Figs. 1 and 2 are the dependences of X and Y on the wave number α for the first two spectrum modes at Re = 10^3. The displacement boundary-layer thickness δ_* is taken as the characteristic scale. The appropriate spectrum numbers of the modes are indicated by the numbers. It is seen from Fig. 1 that such wave numbers α_* exist for which X = 1 and the given spectrum mode vanishes for $\alpha<\alpha_*$ since the physical conditions on the outer limit of the boundary-layer are not satisfied for $X>1$. As the spectrum number n increases, the corresponding limiting wave numbers α_{*n} increase, i.e., $\alpha_{*n-1}<\alpha_{*n}$. Therefore, a finite number of spectrum modes exists for each fixed wave number α. As $\mathrm{Re}\to0$ the wave numbers $\alpha_{*n}\to\infty$ since no discrete spectrum exists in a fluid at rest in a half-space. And, conversely, as the Reynolds number increases α_{*n} decrease, i.e., the mode number grows with the increase in Re for a fixed wave number α.

The first spectrum mode at which flow instability in the boundary layer is realized for Reynolds numbers greater than the critical, turns out to be most dangerous. The critical values of the Reynolds and wave numbers were computed and their values are $\mathrm{Re}=519$, $\alpha=0.304$.

The author is grateful to M. A. Gol'dshtik and V. N. Shtern for useful discussions of the results of the research.

LITERATURE CITED

1. V. N. Shtern, "Small-perturbations spectrum of plane Couette flow," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1969).
2. M. A. Gol'dshtik, V. A. Sapozhnikov, and V. N. Shtern, "Local properties of the hydrodynamic stability problem," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1970).
3. A. M. Sagalakov and V. N. Shtern, "Stability of plane-parallel magnetohydrodynamic flows in a transverse magnetic field," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1970).
4. A. G. Sleptsov, "Eigenvalue estimates in a boundary-layer stability investigation," in: Numerical Methods of Mechanics of a Continuous Medium [in Russian], Vol. 5, No. 5, Novosibirsk (1974).
5. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Part. 1, Nauka, Moscow (1965).
6. I. P. Suprunenko, "Stability of jet flows," Izv. Akad. Nauk SSSR, Mekh., No. 4 (1965).
7. H. Schlichting, Boundary Layer Theory, McGraw-Hill (1968).
8. V. A. Sapozhnikov, "Solution of an eigenvalue problem for ordinary differential equations by the factorization method,: in: Transactions of the All-Union Seminar on Numerical Methods in Viscous Fluid Mechanics, Kanev, 1968 [in Russian], Vol 2, Nauka, Novosibirsk (1969).